Received: *April 1, 1998,* Accepted: *April 3, 1998*

Instanton effects in three-dimensional supersymmetric gauge theories with matter

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Abstract: Using standard field theory techniques we compute perturbative and instanton contributions to the Coulomb branch of three-dimensional supersymmetric QCD with $N = 2$ and $N = 4$ supersymmetry and gauge group $SU(2)$. For the $N = 4$ theory with one massless flavor, we confirm the proposal of Seiberg and Witten that the Coulomb branch is the double-cover of the centered moduli space of two BPS monopoles constructed by Atiyah and Hitchin. Introducing a hypermultiplet mass term, we show that the asymptotic metric on the Coulomb branch coincides with the metric on Dancer's deformation of the monopole moduli space. For the $N = 2$ theory with N_f flavors, we compute the one-loop corrections to the metric and complex structure on the Coulomb branch. We then determine the superpotential including one-loop effects around the instanton background. These calculations provide an explicit check of several results previously obtained by symmetry and holomorphy arguments.

Keywords: Solitons Monopoles and Instantons, Nonperturbative Effects, Extended Supersymmetry.

Contents

1. Introduction

Recently several exact results have been proposed for supersymmetric gauge theories in three dimensions (3D). The Coulomb branches of $N = 4$ theories (theories with eight supercharges) were examined in [1, 2] from a field theory perspective where a connection to monopole moduli spaces was revealed. The arguments presented in these papers rely on certain assumptions about the strong coupling behaviour of these theories which are motivated by input from string theory [3]. These theories were further studied using D-brane technology [4] where an alternative derivation of these results was presented. For $N = 4$ supersymmetric Yang-Mills theory (SYM) without matter, Seiberg and Witten (SW) proposed [1] that the Coulomb branch of the theory is the centered moduli space of two BPS monopoles constructed by Atiyah and Hitchin (AH) [5]. This correspondence was confirmed in [6] by performing explicit one-loop calculations in the vacuum and one-instanton sectors. In fact, as explained in [6], the symmetries of the $N = 4$ theory are so restrictive that the exact form of the metric on the Coulomb branch is uniquely determined by this weak-coupling data. In this sense the calculations presented in [6] can be considered as a direct proof of the SW proposal.

In the present paper we will generalise these calculations to $N = 4$ theories coupled to matter. In particular, for the theory with one massless flavor, SW have proposed that the Coulomb branch is the double-cover of the AH manifold. In the following we confirm this by performing a two-instanton calculation. SW further claimed that the mass parameter of a single hypermultiplet plays the role of Dancer's deformation parameter of this manifold [7]. We show that, at least in perturbation theory, this is indeed the case.

Theories with $N = 2$ supersymmetry (four supercharges) dynamically generate a superpotential on the Coulomb branch. This was calculated for the pure $SU(2)$ theory many years ago in [8]. More recently, proposals for the exact superpotential have been made for a wide range of $N = 2$ theories [9, 10]. In this paper, we calculate the one-instanton contributions to the superpotential for $SU(2)$ theories with matter and confirm the corresponding predictions of [9, 10]. We also compute the one-loop correction to the Kähler metric and complex structure for these models. To show explicitly that the instanton-generated superpotential is holomorphic, an effect discovered in [6] turns out to be crucial. In that paper it was shown in the context of the $N = 4$ theory, that the one-loop determinants arising from integration over bose and fermi fluctuations in the instanton background do not cancel exactly. In the following we will show that, in the $N = 2$ theory, the corresponding residual factor precisely reflects the fact that the superpotential is holomorphic with respect to the one-loop corrected complex structure. We further show how the renormalization group decoupling of massive hypermultiplets is manifest, both in perturbation theory and in the instanton calculus. This enables us to flow from the $N = 4$ to the $N = 2$ model, or from N_f to $N_f - 1$ flavors, by taking an infinite mass limit.

The paper is organised as follows. In section 2 we discuss classical aspects of the models we consider, paying special attention to the different possible mass terms and global symmetries. Section 3 is devoted to perturbative effects. In particular, we show that the perturbative metric on the Coulomb branch of $N = 4$ SYM with a single massive hypermultiplet is indeed given by the asymptotic metric for the Dancer spaces, in agreement with [1]. For the $N = 2$ model, we determine the one-loop correction to the Kähler metric and the complex structure. Some details of these perturbative calculation are presented in an appendix. In section 4 we develop the instanton calculus in theories with arbitrary matter content, discussing the zero mode structure, the collective coordinate measure and the one-loop correction which comes from Gaussian integration over fluctuations around the background of the instanton. In section 5 we restrict ourselves to the $N = 4$ theory. For a single massless hypermultiplet in the fundamental representation, we confirm that the Coulomb branch is the double cover of the AH space. This involves an integration over the relative moduli space of two three-dimensional instantons, which is itself the AH manifold! Finally, in section 6, we consider the $N = 2$ model. We compute the superpotential up to one-loop around the background of an instanton and compare our calculations with the proposals of [8, 9, 10].

2. Fields and symmetries

In this section we discuss the classical theories. In the following, N stands for the number of real two-component Majorana supercharges in 3D theories, whereas $\mathcal{N} = N/2$ denotes the number of complex two-component supercharges which is the usual counting for four dimensional (4D) theories. Three-dimensional theories with $N = 2$ and

$N = 4$ SUSY can be obtained by dimensional reduction of the minimal supersymmetric theories theories in four and six dimensions respectively. As we discuss below, it is also possible to include additional mass terms which have no counterpart in higher dimensions. Unless otherwise stated, the notation is as in [6] (e.g. $\underset{\sim}{X} = X^a \tau^a/2$ denotes adjoint-valued fields).

The multiplets of interest of $N = 2$ SUSY in 3D are obtained by dimensional reduction of $\mathcal{N} = 1$ gauge and chiral multiplets in 4D. The 3D gauge multiplet consists of the gauge field, $\underset{\sim}{A}_\mu$, one two-component Dirac spinor, $\underset{\sim}{\lambda}$, and a real scalar field, $\underset{\sim}{\phi}$, which comes from the component of the 4D gauge field in the reduced direction. The dimensional reduction of the 4D chiral multiplet is the so-called 'half-hypermultiplet' which consists of a single complex scalar and a single Dirac fermion. The $N = 2$ SUSY algebra inherits the chiral R-symmetry of the four-dimensional $\mathcal{N} = 1$ theory, which we will denote $U(1)_N$.

Multiplets of $N = 4$ supersymmetry in three dimensions are obtained by combining $N = 2$ multiplets. The $N = 4$ gauge multiplet consists of the $N = 2$ gauge multiplet introduced above, together with an adjoint half-hypermultiplet which includes a complex scalar, $\underset{\sim}{A}$, and a second 3D Dirac spinor, $\underset{\sim}{\psi}$. The action for $N = 4$ SYM theory with gauge group $SU(2)$ is[1]

$$\begin{aligned} S_{VM} = \frac{2\pi}{e^2}\int \mathrm{d}^3x\ \mathrm{Tr}\ \{ & -\tfrac{1}{2}\underset{\sim}{v}_{\mu\nu}\underset{\sim}{v}^{\mu\nu} + D_\mu \underset{\sim}{\phi} D^\mu \underset{\sim}{\phi} + 2i\,\bar{\underset{\sim}{\lambda}}\, \not{D}\, \underset{\sim}{\lambda} + 2\,\bar{\underset{\sim}{\lambda}}[\underset{\sim}{\phi},\, \underset{\sim}{\lambda}] \\ & +2D_\mu\, \underset{\sim}{A}^\dagger D^\mu\, \underset{\sim}{A} + 2i\,\bar{\underset{\sim}{\psi}}\, \not{D}\, \underset{\sim}{\psi} + [\, \underset{\sim}{A},\, \underset{\sim}{A}^\dagger]^2 + 2|[\underset{\sim}{\phi},\, \underset{\sim}{A}]|^2 \\ & +2\sqrt{2}i([\, \underset{\sim}{A}^\dagger,\, \underset{\sim}{\psi}]\, \underset{\sim}{\lambda} + \, \bar{\underset{\sim}{\lambda}}[\, \underset{\sim}{A},\, \bar{\underset{\sim}{\psi}}]) + 2\,\bar{\underset{\sim}{\psi}}[\underset{\sim}{\phi},\, \underset{\sim}{\psi}]\}\,. \end{aligned} \tag{2.1}$$

Here, the first line taken in isolation is the action of the $N = 2$ SYM theory. The second and third lines provide the gauge couplings, Yukawa couplings and potential terms for the adjoint half-hypermutiplet required to complete the $N = 4$ theory.

$N = 4$ SYM is the dimensional reduction of the minimal supersymmetric theory in six dimensions. The action (2.1) has an $SU(2)_R \times SU(2)_N$ global R-symmetry group. The $SU(2)_N$ is an extension of the four-dimensional chiral $U(1)_N$, and corresponds to rotations in the three reduced directions of the six-dimensional (6D) theory. The $SU(2)_R$ is already present in the 6D theory. The bosons $\underset{\sim}{A} = (\underset{\sim}{\phi}_1 + i\underset{\sim}{\phi}_2)/\sqrt{2}$ and $\underset{\sim}{\phi} = \underset{\sim}{\phi}_3$ combine to form a triplet under $SU(2)_N$ which we denote[2] $\vec{\underset{\sim}{\phi}} = (\underset{\sim}{\phi}_1, \underset{\sim}{\phi}_2, \underset{\sim}{\phi}_3)$. These scalars are singlets of $SU(2)_R$. The fermions $\underset{\sim}{\lambda}$ and $\underset{\sim}{\psi}$ form a doublet under $SU(2)_R$. They also transform under $SU(2)_N$ (see e.g. [6]). The potential has flat directions, allowing the scalars to acquire a VEV. By an $SU(2)_N$ rotation, we can choose the vacuum to be $\langle\, \underset{\sim}{A}\rangle = 0, \langle \underset{\sim}{\phi}\rangle = \sqrt{2}\mathrm{v}\tau^3/2$. The resulting moduli space of vacua is known as the Coulomb branch of the theory. Along these directions the $SU(2)$ gauge group is broken down to $U(1)$ by the the adjoint Higgs mechanism. Gauge fields

[1] We have denoted $|[\phi, A]|^2 = [\phi, A^\dagger][\phi, A]$, and fermion multiplication as e.g. $\lambda\psi \equiv -i\lambda^T\gamma_0\psi$.

[2] In the following, the vector notation $\vec{X} = (X_1, X_2, X_3)$ always denotes a vector of $SU(2)_N$

of the unbroken $U(1)$ subgroup remain massless, while the remaining gauge bosons receive a mass $M_W = \sqrt{2}\mathrm{v}$. On the Coulomb branch the $SU(2)_N$ symmetry is broken to an abelian subgroup, which is denoted $U(1)_N$ as in [1]. This is the same $U(1)_N$ as exists in the $N = 2$ theory.

One can also add matter couplings to (2.1) in a way that preserves either the full $N = 4$ supersymmetry or an $N = 2$ subalgebra. A hypermultiplet of $N = 4$ SUSY consists of two half-hypermultiplets transforming in conjugate representations of the gauge group. Although any number of half-hypermultiplets preserves $N = 2$ supersymmetry, for complex representations of the gauge group the theory suffers a Z_2 anomaly and Chern-Simons terms are dynamically generated unless the half-hypermultiplets are paired to form hypermultiplets. We will not consider the anomalous case. In the following we will introduce N_f hypermultiplets in the fundamental representation of $SU(2)$. Each hypermultiplet contains two complex scalars, q_i and $\tilde{q}_i$, and two Dirac spinors, ψ_i and $\tilde{\psi}_i$, $i = 1, .., N_f$. As the fundamental representation of $SU(2)$ is pseudo-real, all fields transform in the same representation.

The hypermultiplet action respecting $N = 2$ supersymmetry is given by

$$\begin{aligned}\frac{e^2}{2\pi} S_{HM} = \int \mathrm{d}^3x \, \{ & D_\mu q_i^\dagger D^\mu q_i + D_\mu \tilde{q}_i^\dagger D^\mu \tilde{q}_i + i\bar{\psi}_i \not{D} \psi_i + i\bar{\tilde{\psi}}_i \not{D} \tilde{\psi}_i \\ & + i\sqrt{2}(q_i^\dagger \underset{\sim}{\lambda} \psi_i - \bar{\psi}_i \bar{\underset{\sim}{\lambda}} q_i) + i\sqrt{2}(\tilde{q}_i \bar{\underset{\sim}{\lambda}} \bar{\tilde{\psi}}_i - \tilde{\psi}_i \underset{\sim}{\lambda} \tilde{q}_i^\dagger) \\ & + \bar{\psi}_i \underset{\sim}{\phi} \psi_i + \bar{\tilde{\psi}}_i \underset{\sim}{\phi} \tilde{\psi}_i - \tilde{q}_i^\dagger \underset{\sim}{\phi}^2 \tilde{q}_i - q_i^\dagger \underset{\sim}{\phi}^2 q_i - \tfrac{1}{8}(q_i^\dagger \tau^a q_i - \tilde{q}_i \tau^a \tilde{q}_i^\dagger)^2 \} \, . \end{aligned} \tag{2.2}$$

One can also introduce a holomorphic superpotential, $W(\underset{\sim}{A}, q_i, \tilde{q}_i)$, while preserving $N = 2$ supersymmetry. In order to increase the number of supersymmetries to $N = 4$, we must take $W = \sqrt{2}\tilde{q}_i \underset{\sim}{A} q_i$ with the corresponding action given by

$$\begin{aligned}\frac{e^2}{2\pi} S_W = \int \mathrm{d}^3x \Big\{ & -\frac{1}{2} \frac{\partial^2 W}{\partial\phi_B \partial\phi_C} \psi_B \psi_C + \text{h.c.} \\ & - \sum_B \left| \frac{\partial W}{\partial \phi_B} \right|^2 + q_i^\dagger [\underset{\sim}{A}^\dagger, \underset{\sim}{A}] q_i - \tilde{q}_i [\underset{\sim}{A}^\dagger, \underset{\sim}{A}] \tilde{q}_i^\dagger \Big\} \, , \end{aligned} \tag{2.3}$$

where we have defined $\phi_B = \{\underset{\sim}{A}, q_i, \tilde{q}_i\}$ and $\psi_B = \{\underset{\sim}{\psi}, \psi_i, \tilde{\psi}_i\}$ for $B = 1, 2, 3$. The final two terms in (2.3) arise from integrating out the auxiliary field associated with the 4D vector multiplet ("D-terms"). They ensure the bosonic potential is invariant under $SU(2)_N$. In this case the scalars $(q, \tilde{q}^\dagger)$ form a doublet under $SU(2)_R$ and do not transform under $SU(2)_N$. The hypermultiplet fermions are singlets under $SU(2)_R$ but do transform under $SU(2)_N$, see [11].

Finally, we discuss mass terms. Four-dimensional $\mathcal{N} = 2$ theories permit a complex mass parameter, m, for each hypermultiplet. This is most conveniently described by changing the superpotential to $W = \sqrt{2}\tilde{q}_i \underset{\sim}{A} q_i + m\tilde{q}_i q_i$. Three-dimensional $N = 4$ theories allow for an extra real mass parameter, $\tilde{m}$, corresponding to the mass of 3D Dirac fermions. Unlike the complex mass, it cannot be written as part of the superpotential. However the real mass can be introduced in a manifestly supersymmetric way by gauging a subgroup of the global flavor symmetry and then freezing the gauge multiplet to a background scalar expectation value. In a real basis the three masses $\vec{m} = (\mathrm{Re}(m), \mathrm{Im}(m), \tilde{m})$ transform as a vector under $SU(2)_N$. In component form we have,

$$\frac{e^2}{2\pi} S_m = \int \mathrm{d}^3 x \, \{ -m_i \psi_i \tilde{\psi}_i - \bar{m}_i \bar{\psi}_i \bar{\tilde{\psi}}_i - \tilde{m}(\bar{\psi}_i \psi_i + \bar{\tilde{\psi}}_i \tilde{\psi}_i) \\ - |\vec{m}|^2 (\tilde{q}_i \tilde{q}_i^\dagger + q_i q_i^\dagger) - \sqrt{2}(\tilde{q}_i \vec{m} \cdot \underset{\sim}{\vec{\phi}} \tilde{q}_i^\dagger + q_i^\dagger \vec{m} \cdot \underset{\sim}{\vec{\phi}} q^i) \} \,. \tag{2.4}$$

We will also consider the mass deformed $N = 4$ obtained by giving a mass, $\vec{M}$, to the adjoint half-hypermultiplet

$$\frac{e^2}{2\pi} S_M = \mathrm{Tr} \int \mathrm{d}^3 x \{ -M \underset{\sim}{\psi} \underset{\sim}{\psi} - \bar{M} \underset{\sim}{\bar{\psi}} \underset{\sim}{\bar{\psi}} - \tilde{M} \underset{\sim}{\bar{\psi}} \underset{\sim}{\psi} - |\vec{M}|^2 \underset{\sim}{A}^\dagger \underset{\sim}{A} \} \,. \tag{2.5}$$

Taking an infinite mass limit for the adjoint half-hypermultiplet, the theory flows to $N = 2$ SYM. Similarly, on taking the mass of a single fundamental hypermultiplet to infinity, the theory with N_f flavors flows to the theory with $N_f - 1$ flavors. This decoupling provides a useful check on the perturbative and instanton calculations presented below.

3. Perturbation theory and the effective action

In three dimensions the gauge coupling has the dimensions of mass and 3D gauge theories are therefore super-renormalisable. The perturbation series is organised in powers of the dimensionless quantity e^2/M_W. In the following, we will consider the finite renormalisation of the coupling constant at one-loop. This may be calculated by integrating out high frequency modes to obtain the Wilsonian effective action for the massless fields. An explicit calculation of this effect for the $N = 4$ theory without matter was presented in [6] (In particular, see Appendix B of this reference.). In this section, we present the generalization of these results to include additional matter multiplets with and without mass terms. We then use the one-loop renormalization of the coupling to determine the asymptotic behaviour of the metric on the Coulomb branch. We initially restrict ourselves to $N = 4$ model. The case of $N = 4$ with one massive flavor will be discussed in detail. The $N = 2$ case is treated at the end of the section.

Consider massless hypermultiplets, which can be in the adjoint or fundamental representations of the gauge group. Let N_a and N_f denote the number of massless hypermultiplets in these representations respectively. A straightforward modification of the calculation given in [6] shows that the one-loop renormalisation to the coupling constant is given by the replacement

$$\frac{2\pi}{e^2} \to \frac{2\pi}{e^2} \left(1 - \frac{2 - N_f - 2N_a}{S_{\mathrm{cl}}} \right) , \tag{3.1}$$

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where, for later convenience, we have expressed the answer in terms of $S_{\rm cl} \equiv 8\pi^2 M_W/e^2$ which is equal to the instanton action. Some details of this result and its generalization to include hypermultiplet masses are given in the appendix. Here we concentrate on the case of one massive fundamental hypermultiplet. The corresponding renormalization is

$$\frac{2\pi}{e^2} \to \frac{2\pi}{e^2}\left(1 - \frac{2}{S_{\rm cl}} + \frac{e^2}{2^5\pi^2}\sum_{\epsilon=1,2} |\vec{m} + (-1)^{\epsilon}\vec{\rm v}/\sqrt{2}|^{-1}\right). \tag{3.2}$$

The above expression correctly goes over to the $N_f = 0$ and $N_f = 1$ results in the decoupling limit $|\vec{m}| \to \infty$ and the massless limit $|\vec{m}| \to 0$ respectively.

To see how these results determine the one-loop metric on the Coulomb branch, we must first perform a duality transformation, eliminating the massless photon in favour of a periodic scalar. We follow closely the discussion and notation in [6]. At the classical level, the bosonic low-energy action is simply given by the free massless expression

$$S_{\rm B} = \frac{2\pi}{e^2}\int {\rm d}^3x \left[\tfrac{1}{4}v_{\mu\nu}v^{\mu\nu} + \tfrac{1}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi}\right], \tag{3.3}$$

where $v_{\mu\nu} = {\rm Tr}(\underset{\sim}{v}_{\mu\nu}\tau^3)$ with similar definitions for the massless scalar fields. In three dimensions, the photon is dual to a scalar that serves as a Lagrange multiplier for the Bianchi identity. Hence we add to the action a term,

$$S_{\rm S} = \frac{i}{8\pi}\int {\rm d}^3x\, \sigma\varepsilon^{\mu\nu\rho}\partial_\mu v_{\nu\rho}\,. \tag{3.4}$$

In a topologically trivial background we may integrate this term by parts, disregarding the surface term, to find that the action only depends on σ through its derivatives and therefore the theory has a trivial symmetry $\sigma \to \sigma + c$ for constant c. On the other hand, in the presence of an instanton, the surface term is non-zero. In fact, the normalization of (3.4) was chosen so that $S_{\rm S} = -i\sigma$ in the background of an instanton of unit magnetic charge. It follows that σ is a periodic variable with period 2π. Integrating out the abelian field strength then yields the bosonic effective action,

$$S_{\rm B} = \frac{2\pi}{e^2}\int {\rm d}^3x\ \tfrac{1}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} + \frac{2e^2}{\pi(8\pi)^2}\int {\rm d}^3x\ \tfrac{1}{2}\partial_\mu\sigma\partial^\mu\sigma\,. \tag{3.5}$$

At one-loop, the above action is modified by the replacement (3.1). Although there are other one-loop effects, as we explain below the full effective action at one-loop is uniquely determined by the coupling constant renormalization.

On the Coulomb branch, the massless fields consist of four real scalar fields and two Dirac fermions, which can be seen as four real Majorana fermions. The most general form for the low-energy effective action up to two derivatives and four fermi terms takes the form of a supersymmetric sigma model with a four dimensional target manifold, $\mathcal{M}$. Due to $N = 4$ supersymmetry, the metric on $\mathcal{M}$ is hyper-Kähler [12]. In some set of real coordinates, X^a, and their supersymmetric Majorana partners, Ω^a, $a = 1,..,4$, the action reads

$$S_{\rm eff} = K\int {\rm d}^3x\ \left\{\tfrac{1}{2}g_{ab}(X)\ \left[\partial_\mu X^a\partial^\mu X^b + \bar{\Omega}^a \not{D}\Omega^b\right] + \tfrac{1}{12}R_{abcd}(\bar{\Omega}^a\Omega^c)(\bar{\Omega}^b\Omega^d)\right\}, \tag{3.6}$$

where the derivative $\not{D}$ is covariant with respect to the hyper-Kähler metric g_{ab}, and R_{abcd} denotes the Riemann tensor on $\mathcal{M}$. For later convenience we have included an overall normalisation constant K whose value will be fixed below.

In addition to the hyper-Kähler property, the part of the metric derived solely from perturbation theory must have a $U(1)$ isometry corresponding to the freedom to shift σ by a constant. As mentioned above, this symmetry is broken by instantons. However it is respected by all perturbative corrections. Any four-dimensional hyper-Kähler metric with such a triholomorphic isometry can be written in the form [13],

$$g_{ab}\mathrm{d}X^a\mathrm{d}X^b = U(\vec{r})\mathrm{d}\vec{r}\cdot\mathrm{d}\vec{r} + 4U^{-1}(\vec{r})(\mathrm{d}X^4 + \vec{w}\cdot\mathrm{d}\vec{r})^2\,, \tag{3.7}$$

where $\vec{r} = (X^1, X^2, X^3)$ and $\vec{w}$ satisfies $\vec{\nabla}\times\vec{w} = -(1/2)\vec{\nabla}U$. Hence we may determine the metric at one-loop by calculating the function U to this order. This may be accomplished by comparing the general forms (3.6) and (3.7) with our classical effective action (3.5) suplemented by the one-loop replacement (3.2). From this comparison we may deduce the relationship between the fields, $\vec{\phi}$ and σ and the coordinates, X^a, on the manifold correct to one loop. We fix the normalisation by requiring the classical metric to have $U = 1$ and, (for $N_f = 1$), $X^4 = \sigma$. This implies the identification, $\vec{r} = (16\pi^2/e^2)\vec{\phi}$ and $K = 2e^2/2^8\pi^3$ and hence the radial coordinate is given as $r = 2S_{\mathrm{cl}}$. In general these relations will be modified by higher loop corrections.

Using these identifications, the function U is given at one-loop as,

$$U = 1 - \frac{4}{r} + \frac{1}{|\vec{\mu}-\vec{r}|} + \frac{1}{|\vec{\mu}+\vec{r}|}\,, \tag{3.8}$$

where $\vec{\mu} = (2^5\pi^2/e^2)\vec{m}$. In [1], Seiberg and Witten proposed that the the Coulomb branch of the $N = 4$ theory with one massive hypermultiplet coincides with the three-parameter family of deformations $\mathcal{M}(\vec{\lambda})$ of the double cover of the AH manifold discovered by Dancer [7]. The deformation parameter $\vec{\lambda}$ was identified as a multiple of the hypermultiplet mass $\vec{m}$. In fact the metric on Dancer's manifold is known explicitly only in the asymptotic regime. It can be extracted from the $SU(3)$ $(2,1)$ monopole moduli space in the limit that the third (distinct) monopole becomes infinitely massive [14]. It is indeed of the form (3.7) with U given by (3.8) and $\vec{\lambda} = \vec{\mu}$, see eqn. (27) in [15]. This shows that our perturbative calculation is in agreement with the conjecture of Seiberg and Witten.

Finally, we turn to the $N = 2$ model with N_f fundamental hypermultiplets. The massless fields are a single real scalar $\phi = \mathrm{Tr}(\underset{\sim}{\phi}\tau^3)$ and the dual photon σ. At the classical level the bosonic low-energy effective action is,

$$S_{\mathrm{B}} = \frac{2\pi}{e^2}\int \mathrm{d}^3x\ \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{2e^2}{\pi(8\pi)^2}\int \mathrm{d}^3x\ \tfrac{1}{2}\partial_\mu\sigma\partial^\mu\sigma\,. \tag{3.9}$$

As in the $N = 4$ case, the action is modified at one-loop by a finite renormalization of the gauge coupling. The one-loop correction is determined by the results given in the

Appendix, see (6.10). In the case of massless hypermultiplets,

$$\frac{2\pi}{e^2} \to \frac{2\pi}{e^2}\left(1 - \frac{3-N_f}{S_{\rm cl}}\right). \tag{3.10}$$

As before, we would like to compare the result of our perturbative calculation with the corresponding terms in the most general action allowed by the symmetries of the theory. The terms with two derivatives and their supersymmetric completion must again take the form of a supersymmetric non-linear sigma model. However in the $N = 2$ case the target is a Kähler manifold of complex dimension one. In terms of complex coordinates Z, $\bar{Z}$ and 3D Dirac superpartners Ψ, $\bar{\Psi}$ the effective action is,

$$S_{\rm eff} = L\int \mathrm{d}^3x\,\left\{g_{\bar{Z}Z}\left[\partial_\mu \bar{Z}\partial^\mu Z + \bar{\Psi}\not{D}\Psi\right] + \tfrac{1}{4}R(Z,\bar{Z})\bar{\Psi}^2\Psi^2\right\}. \tag{3.11}$$

The metric is derived from a Kähler potential, $g_{\bar{Z}Z} = \partial_{\bar{Z}}\partial_Z K(Z,\bar{Z})$, and $R(Z,\bar{Z})$ is the non-vanishing component of the Riemann tensor.

The classical effective action (3.9) can trivially be written in the form (3.11) with complex variable $Z = S_{\rm cl} - i\sigma$ and Kähler potential $K = Z\bar{Z}$. This normalization fixes the overall constant in (3.11) as $L = 2e^2/(\pi(8\pi)^2)$. To incorporate the one-loop effect, one must redefine the complex coordinates as [9, 10]

$$Z = S_{\rm cl} - (3-N_f)\log S_{\rm cl} - i\sigma\,. \tag{3.12}$$

The one-loop Kähler metric is $(1-(3-N_f)/S_{\rm cl})^{-1}$ which is defined as an implicit function of $Z+\bar{Z}$ by (3.12). Using the results of the appendix, one easily generalises this to N_f massive flavors. The result for the one-loop corrected complex structure can be written as

$$Z = S_{\rm cl} - i\sigma - 3\log S_{\rm cl} + \tfrac{1}{2}\log\left(\prod_{i=1}^{N_f}\left[\frac{\sqrt{|m_i|^2 + (\tilde{m}_i + \frac{1}{2}M_W)^2} + \tilde{m}_i + \frac{1}{2}M_W}{\sqrt{|m_i|^2 + (\tilde{m}_i - \frac{1}{2}M_W)^2} + \tilde{m}_i - \frac{1}{2}M_W}\right]\right). \tag{3.13}$$

Notice that, on taking the masses of one of the N_f flavors to infinity we correctly recover the result for $N_f - 1$ flavors.

In the $N = 2$ case, it is also possible to generate a holomorphic superpotential $W(Z)$ which preserves supersymmetry. This gives rise to a bosonic potential in the low-energy effective action as well as fermion bilinear terms;

$$S_W = \int \mathrm{d}^3x\,\left\{\left|\frac{\partial W}{\partial Z}\right|^2 + \tfrac{1}{2}\frac{\partial^2 W}{\partial Z^2}\Psi^2 + \tfrac{1}{2}\frac{\partial^2 \bar{W}}{\partial \bar{Z}^2}\bar{\Psi}^2\right\}. \tag{3.14}$$

As mentioned above the perturbative theory has a trivial symmetry under which σ and therefore Z transforms additively. Clearly this implies that the above superpotential cannot be generated at any order in perturbation theory. However instantons break this symmetry and can contribute terms to W such as $\exp(-Z)$ which are periodic in σ with period 2π. Importantly, the instanton-induced superpotential is holomorphic in the superfield Z which implies non-trivial quantum corrections in the instanton background when re-expressed in terms of $S_{\rm cl}$ and σ. In section 5 we will check explicitly that the instanton contribution is holomorphic with respect to the one-loop corrected complex structure.

4. Supersymmetric instantons in 3D

The study of instantons in three-dimensional gauge theories was initiated by Polyakov in the context of quark confinement [16]. The relevant field configurations of finite Euclidean action are the static monopole solutions of (3+1)-dimensional gauge theory. In the supersymmetric theories considered here, the Prasad-Sommerfield limit is automatic and hence the instantons we will consider are BPS monopoles.

Instanton effects in the three-dimensional gauge theories with $N = 4$ supersymmetry were studied in [1, 2, 4]. Explicit computations were performed for the $SU(2)$ theory without matter in [6] and our methods here follow the same strategy. It is also useful for the reader to consult [17]. In the present section we will discuss the zero modes of the instanton and the measure for integration over the collective coordinates, as well as the non-cancelling determinants coming from integration over quadratic fluctuations around the instanton background. In section 5, we will perform a two instanton calculation in the case of one massless hypermultiplet, which confirms the conjecture of Seiberg and Witten that the quantum moduli space of this theory is given by the double cover of the Atiyah-Hitchin manifold.

Instanton effects in the three-dimensional $N = 2$ $SU(2)$ Yang-Mills theory were studied in [8], where it was shown that a superpotential is generated at the one-instanton level. Exact forms for the instanton-generated superpotential for $N = 2$ theories coupled to matter were proposed in [9, 10]. In section 6, we will test these proposals against explicit semiclassical calculations. The relevant formulae for constructing the instanton measure in the $N = 2$ theory are given at the end of this section.

For $N = 4$ SYM, in the absence of matter, the instanton zero modes were discussed in detail in [6, 17]. The Callias index theorem dictates that there are $4k$ bosonic and $4k$ fermionic zero modes, where k is the instanton number. The classical action for the k-instanton is $S_k = kS_{\rm cl}$, where the single-instanton action, $S_{\rm cl} = 8\pi^2 M_W/e^2$ was defined in the previous section. The moduli space of instantons decomposes as $R^3 \times (S^1 \times \tilde{M}_k)/Z_k$. The R^3 and S^1 factors correspond to the spacetime position X_μ, $\mu = 0, 1, 2$, of the 'centre of mass' and to the 'centre of charge' θ of the instanton. We will refer to these four degrees of freedom as the 'centre coordinates'. The relative moduli space $\tilde{M}_k$ is a smooth hyper-Kähler manifold of dimension $4(k-1)$ equipped with a natural metric[3]

[3]For the case $k = 2$ it is a notable coincidence (and also a source of potential confusion) that the relative moduli space of two instantons $\tilde{M}_2$ is the double-cover of the AH manifold which is precisely the conjectured vacuum moduli space of the theory.

$\tilde{g_{ab}}$. Roughly speaking the coordinates on $\tilde{M}_2$, which we denote $Y^a; a = 1, ..., 4(k-1)$, parametrize the relative separations and $U(1)$ charge angles of k instantons. The discrete symmetry Z_k acts on both the S^1 and on relative charges contained within $\tilde{M}_k$. At leading semiclassical order, the path integral measure can be written as an integral over these collective coordinates with an appropriate Jacobian [18]. The additional prefactor which arises from including one-loop effects in the instanton background will be considered below. For the bosonic sector the collective coordinate measure consists of two factors corresponding to the centre and relative coordinates respectively. Explicitly [17] we have $\mathrm{d}\mu_B = \mathrm{d}\bar{\mu}_B \mathrm{d}\tilde{\mu}_B$ with

$$\int \mathrm{d}\bar{\mu}_B = \frac{2^4\pi^2 k^2 M_W}{e^4} \int \mathrm{d}^3 X \int_0^{2\pi} \mathrm{d}\theta \; ; \int \mathrm{d}\tilde{\mu}_B = \frac{1}{(2\pi)^{2(k-1)}} \int \Pi_{a=1}^{4(k-1)} \mathrm{d}Y^a \sqrt{\tilde{g}} \, . \quad (4.1)$$

We have still to divide by the Z_k symmetry factor. We will discuss this in more detail when considering $k = 2$ below.

In the $N = 4$ theory, four of the total of $4k$ adjoint fermion zero modes arise from the action of four supersymmetry generators on the monopole background. The monopole background is invariant under the other SUSY generators. Unlike the remaining fermionic zero modes, these modes are protected from lifting by SUSY. We introduce two two-component Grassmann collective coordinates ξ_α, ξ'_α. In the following we will need the large-distance behaviour of these modes. Consider a k-monopole solution with centre of mass coordinate, X_μ. Then, for $|x - X| \gg M_W^{-1}$, we have [6, 17]

$$\begin{aligned} \lambda_\alpha^{\mathrm{LD}} &= 8\pi k \left(S_{\mathrm{F}}(x - X)\right)_\alpha^{\ \beta} \xi_\beta \\ \psi_\alpha^{\mathrm{LD}} &= 8\pi k \left(S_{\mathrm{F}}(x - X)\right)_\alpha^{\ \beta} \xi'_\beta \, , \end{aligned} \quad (4.2)$$

where $S_{\mathrm{F}}(x) = \gamma^\mu x_\mu / (4\pi |x|^3)$ is the three-dimensional Dirac fermion propagator. These modes are the SUSY partners of the centre coordinates, X_μ and θ. As in [17], we also introduce Grassmann collective coordinates α^a, $a = 1, \ldots 4(k-1)$ which correspond to the remaining $4(k-1)$ adjoint fermion zero modes. These coordinates are Grassmann numbers rather than two-component Grassmann spinors, reflecting the fact that the number of adjoint zero modes is exactly half that of the $N = 8$ case considered in [17]. These modes are not protected by supersymmetry and we will see below that they are lifted in the presence of additional matter multiplets. As for the bosonic zero modes, it is convenient to write the measure for the adjoint fermionic collective coordinates as a product of two factors corresponding to the centre and relative coordinates, $d\mu_F = d\bar{\mu}_F d\tilde{\mu}_F$ with

$$\int \mathrm{d}\bar{\mu}_F = \frac{e^4}{2^8 k^2 \pi^4 M_W^2} \int \mathrm{d}^2\xi \mathrm{d}^2\xi' \; ; \int \mathrm{d}\tilde{\mu}_F = \int \Pi_{a=1}^{4(k-1)} \mathrm{d}\alpha^a \frac{1}{\sqrt{\tilde{g}}} \, , \quad (4.3)$$

where $\mathrm{d}^2\xi = (1/2)\mathrm{d}\xi_1 \mathrm{d}\xi_2$.

So far, we have only considered the vector multiplets. We now include hypermultiplets, with masses as in (2.4). The relevant zero modes are solutions of the Dirac equation for the hypermultiplet fermions in the instanton background. The number of linearly independent solutions is given by the Callias index theorem [20]. Following Weinberg [19], we define the operators,

$$\begin{aligned}\Delta_{-}^{R}(\vec{m}) &= D_{\rm cl}^2 + 2\gamma^\mu \underset{\sim}{B}_\mu + |m|^2 + (\tilde{m} + \underset{\sim}{\phi}_3)^2 \\ \Delta_{+}^{R}(\vec{m}) &= D_{\rm cl}^2 + |m|^2 + (\tilde{m} + \underset{\sim}{\phi}_3)^2 ,\end{aligned} \tag{4.4}$$

where where $\underset{\sim}{B}^\mu_{\rm cl} = (1/2)\epsilon^{\mu\nu\rho}\underset{\sim}{v}^{\rm cl}_{\nu\rho}$ is the magnetic field of the BPS monopole and the superscript R denotes the representation of the gauge group generators appearing in (4.4). In the following $R = A$ denotes the adjoint representation and $R = F$ denotes the fundamental representation. The number of zero modes for a three-dimensional Dirac fermion transforming in the representation R is given by the limit $\alpha^2 \to 0$ of the regularized trace,

$$\mathcal{L}_{\vec{m}}(\alpha^2) = {\rm Tr}\left[\frac{\alpha^2}{\Delta_{-}^{R}(\vec{m}) + \alpha^2} - \frac{\alpha^2}{\Delta_{+}^{R}(\vec{m}) + \alpha^2}\right]. \tag{4.5}$$

For zero complex mass, this trace was evaluated in [9] (see appendix of the second reference). The generalization to non-zero complex mass is straightforward and yields,

$$\mathcal{L}(\alpha^2) = \frac{\alpha^2}{\alpha^2 + |m|^2}\sum_w \frac{(w^2 M_W + w\tilde{m})k}{(\alpha^2 + |m|^2 + (\tilde{m} + wM_W)^2)^{1/2}} , \tag{4.6}$$

where the trace over gauge indices has been exchanged for a sum over weights, w, of the representation R. Putting all masses to zero and specifying the adjoint representation, which has weights $1, 0$ and -1, the number of zero modes is $2k$ for each species of Dirac fermion. As the $N = 4$ vector multiplet contains two such species this result agrees with the counting of adjoint fermion zero modes given above. For each species of massless fermion in the fundamental representation (weights $\frac{1}{2}$ and $-\frac{1}{2}$) the number is reduced to k. In the presence of the complex mass all zero modes are lifted. However, in the presence of only a real mass, the zero mode structure depends on the relative values of $\tilde{m}$ and M_W. For $M_W > 2|\tilde{m}|$ there are k zero modes, and for $M_W < 2|\tilde{m}|$ there are none [20]. For $M_W = 2\tilde{m}$, the zero modes become non-renormalisable. This corresponds to the point on the Coulomb branch where the quarks are classically massless.

Just as for the other modes, we can introduce collective coordinates for the hypermultiplet zero modes. We focus on fundamental hypermultiplets with all masses set to zero and hence each species of Dirac fermion has k zero modes. For each hypermultiplet with Dirac fermions ψ_i and $\tilde{\psi}_i$, we can expand the zero mode solution of the Dirac equation in a complex orthonormal basis with Grassmann coefficients λ^A, $A = 1, ..., k$

$$\psi_i = \rho_i^A \lambda^A \qquad \tilde{\psi}_i = \tilde{\rho}_i^A \lambda^A , \tag{4.7}$$

where $i = 1, ..., N_f$ and

$$\int {\rm d}^3x\, \lambda^{A\dagger}\lambda^B = \delta^{AB} . \tag{4.8}$$

Note that the charge conjugate fermions have no zero mode solutions, hence we set $\bar{\psi}_i = \bar{\tilde{\psi}}_i = 0$. Using this normalization, the collective coordinate measure for the zero modes of N_f fundamental hypermultiplets is

$$\int \mathrm{d}\mu_f = \Big(\frac{e^2}{2\pi}\Big)^{kN_f} \int \Pi_{i=1}^{N_f} \Pi_{A=1}^{k}\, \mathrm{d}\rho_i^A \mathrm{d}\tilde{\rho}_i^A \,. \tag{4.9}$$

As mentioned above, the four adjoint fermion zero modes parametrized by ξ_α, ξ'_α are protected by supersymmetry and cannot be lifted. However this is not the case for either the remaining adjoint fermion zero modes parametrized by α^a or the hypermultiplet fermi zero modes with coordinates ρ_i^A, $\tilde{\rho}_i^A$. As the hypermultiplet and vector multiplet fermions have opposite charge under the unbroken R-symmetry denoted $U(1)_N$, Seiberg and Witten [1] suggested that these modes can be lifted in pairs. In the following we will exhibit this lifting explicitly. In fact a similar effect is known to occur for the case of a single adjoint hypermultiplet, which corresponds to the $N = 8$ model discussed in [17]. In that case the lifting of modes was due to the presence of a SUSY- and $U(1)_N$-invariant Grassmann quadrilinear term in the action of the instanton. The quadrilinear term was found by dimensional reduction of the collective coordinate Lagrangian for the low-energy dynamics of BPS monopoles in $(3+1)$-dimensional $\mathcal{N} = 4$ SUSY YM. This procedure yielded a quadrilinear term proportional to the Riemann tensor on the instanton moduli space.

In the present case of fundamental hypermultiplets, a similar term can be deduced by dimensional reduction of the corresponding theory in four dimensions. The relevant collective coordinate Lagrangian for the low-energy dynamics of BPS monopoles in four-dimensional $\mathcal{N} = 2$ SQCD has been given in [22, 23, 24]. It contains a term bilinear in α^a and in ρ_i^A ($\tilde{\rho}_i^A$). After reducing to three dimensions, the resulting instanton action is [4]

$$\tilde{S} = kS_{\rm cl} - ik\sigma - \frac{2\pi}{e^2}\left(\frac{1}{4} F_{ab}^{AB} \alpha^a \alpha^b (\rho_i^A \rho_i^B + \tilde{\rho}_i^A \tilde{\rho}_i^B)\right). \tag{4.10}$$

Here F_{ab}^{AB} is the self-dual curvature tensor of a certain $O(k)$ bundle over the instanton moduli space. In the case $k = 2$, which is treated in detail in the next section, explicit formulae for this tensor are given in [25]. From a three-dimensional perspective, the quadrilinear term comes from the Yukawa terms in the action. The field equations set the scalars of the vector and hypermultiplet to be quadratic in the fermionic collective coordinates, yielding Yukawa terms quartic in these coordinates.

As in any semiclassical instanton calculation, to complete the specification of the measure we must also consider the contribution of non-zero modes. Often in supersymmetric gauge theories these contributions cancel between bose and fermi degrees of freedom. However, as explained in [6], this cancellation does not occur in three-dimensional

[4]To compare with the results in [24], one must perform an $SU(2)_N$ rotation. This acts on the vector multiplet scalars, rotating the vev v_1 into v_3, and also on the hypermultiplet fermions. The net effect of this is that our ρ^A and λ^A are the same as in [24]. This fixes the normalisation of the F term in (4.10).

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theories with $N = 4$ SUSY, because of the spectral asymmetry of the Dirac operator in a monopole background. In the case of $N = 4$ SYM considered in [6] the residual factor involves the ratio of the determinants of the operators $\Delta_\pm = \Delta^A_\pm(\vec{m} = 0)$,

$$R_V = \left[\frac{\det(\Delta_+)}{\det'(\Delta_-)}\right]^{\frac{1}{2}} = (2M_W)^{2k} \,. \tag{4.11}$$

Here Δ_+ is positive and has no zero modes, while Δ_- has $2k$ zero modes [19]. The prime in (4.11) denotes the removal of these zero modes.

In the present case, the contribution of the vector multiplet is given by (4.11), and there are additional one-loop factors in the measure for each hypermultiplet. Including scalar and fermion contributions from each fundamental hypermultiplet, the Gaussian integral over quadratic fluctuations around the instanton yields the factor

$$R_H = \prod_{i=1}^{N_f} \left[\frac{\det'\Delta^F_-(\vec{m}_i)}{\det\Delta^F_+(\vec{m}_i)}\right]^{\frac{1}{2}} . \tag{4.12}$$

Using identical manipulations to those given in [6], each term in this product may be related to an integral over the regularized trace, (4.6)

$$R_H = \prod_{i=1}^{N_f} \left[\lim_{\alpha\to 0} \left(\alpha^{n_i} \exp[\int_\alpha^\infty \frac{d\mu}{\mu} \mathcal{L}_{\vec{m}_i}(\mu)]\right)^{-\frac{1}{2}}\right] , \tag{4.13}$$

where n_i is the number of zero modes of the i'th hypermultiplet.

For zero masses $\vec{m}_i = 0$, the Callias index theorem tells us that $n_i = k$ and hence we have a one-loop factor

$$R_H = (M_W)^{-kN_f} \,. \tag{4.14}$$

In the next section, we consider one massless fundamental hypermultiplet with (4.14) the relevant formula. On the other hand with all complex masses chosen to be non-zero, $n_i = 0$ for each i and we obtain,

$$R_H = \prod_{i=1}^{N_f} \left[\frac{\sqrt{|m_i|^2 + (\tilde{m}_i + \frac{1}{2}M_W)^2} + \tilde{m}_i + \frac{1}{2}M_W}{\sqrt{|m_i|^2 + (\tilde{m}_i - \frac{1}{2}M_W)^2} + \tilde{m}_i - \frac{1}{2}M_W}\right]^{-k/2} . \tag{4.15}$$

Notice that sending one of the masses to infinity reduces the corresponding factor to unity and the N_f flavor theory flows to the $N_f - 1$ flavor theory.

As in [21, 17], one can also consider a theory with $N = 8$ supersymmetry by adding a single massless adjoint hypermultiplet to $N = 4$ SYM. In this case, the adjoint hypermultiplet yields a one-loop factor R_V^{-1} and cancels the determinant from the vector multiplet. Alternatively, one can consider mass deformed $N = 8$ where the adjoint hypermultiplet is given a mass. For a non-zero complex mass, the ratio of determinants for a massive adjoint hypermultiplet is,

$$R_A = \left[\frac{\sqrt{|M|^2 + (\tilde{M}_i + M_W)^2} + \tilde{M}_i + M_W}{\sqrt{|M|^2 + (\tilde{M}_i - M_W)^2} + \tilde{M}_i - M_W}\right]^{-k} . \tag{4.16}$$

Finally we will briefly state the modifications required to obtain the correct instanton measure in the $N = 2$ theory with matter. The main difference is that the number of adjoint fermion zero modes in the instanton background is halved. In the k-instanton sector there are now $2k$ adjoint fermion zero modes of which only two are protected by supersymmetry. The corresponding collective coordinate measure is given by $d\nu_F = d\bar{\nu}_F d\tilde{\nu}_F$ with

$$\int \mathrm{d}\bar{\nu}_F = \frac{e^2}{2^4 k\pi^2 M_W} \int \mathrm{d}^2\xi \; ; \int \mathrm{d}\tilde{\nu}_F = \int \Pi_{a=1}^{2(k-1)} \mathrm{d}\alpha^a \tilde{g}^{-\frac{1}{4}} \, . \tag{4.17}$$

Similarly the resulting one-loop determinant factor differs from that of the $N = 4$ theory by the contribution of the additional massless adjoint half-hypermultiplet which is present in the latter theory. Explicitly R_V of equation (4.11) is replaced by;

$$S_V = \left[\frac{\det(\Delta_+)}{\det'(\Delta_-)}\right]^{3/4} = (2M_W)^{3k} \, . \tag{4.18}$$

It is also likely that the exact form of the Grassmann quadrilinear term in the multi-instanton action (4.10) is different in the $N = 2$ theory. However the form of this term is constrained by invariance under the $U(1)_N$ symmetry of the $N = 2$ theory. As mentioned above, zero modes of the vector multiplet fermions have opposite $U(1)_N$ charge to those of the hypermultiplet fermions. The multi-instanton action will therefore have the general form,

$$\tilde{S}_{N=2} = kS_{\mathrm{cl}} - ik\sigma + O(\alpha^2\rho^2) \, . \tag{4.19}$$

In fact this symmetry argument will be enough to show that there are no instanton corrections to the superpotential in the $N = 2$ theory with massless hypermultiplets.

It is possible to flow from the $N = 4$ to the $N = 2$ theory by adjusting the mass of the extra adjoint half-hypermultiplet. For non-zero complex mass, this augments the ratio of determinants (4.18) by the factor $R_A^{1/2}$.

5. Instanton effects in $N = 4$ theories

We will compute the leading order exponential corrections to the Riemann tensor of the Coulomb branch metric (3.6). When combined with the perturbative result and the constraints of the hyper-Kähler condition this will prove sufficient to determine the metric fully. The Riemann tensor appears in the low energy effective action (3.6) as the coefficient of a four-fermion vertex. To determine the instanton contribution to this vertex we therefore evaluate the large-distance behaviour of the four-fermi correlator,

$$G^{(4)}(x_1, x_2, x_3, x_4) = \langle \lambda_\alpha(x_1)\lambda_\beta(x_2)\psi_\gamma(x_3)\psi_\delta(x_4)\rangle \; , \tag{5.1}$$

where the fermion fields take their zero-mode values in the instanton background. Collecting the various factors from the previous section the leading semiclassical contribution to this correlator is

$$G^{(4)}(x_1,x_2,x_3,x_4) = \int \mathrm{d}\mu_B \mathrm{d}\mu_F \mathrm{d}\mu_f \lambda^{\mathrm{LD}}_\alpha(x_1)\lambda^{\mathrm{LD}}_\beta(x_2)\psi^{\mathrm{LD}}_\gamma(x_3)\psi^{\mathrm{LD}}_\delta(x_4) \times R_V R_H \exp\left(-\tilde{S}\right). \tag{5.2}$$

Definitions of the various ingredients in this formula can be found in equations (4.1), (4.2), (4.3), (4.9), (4.10), (4.11) and (4.14).

For (5.2) to yield a non-zero answer it is necessary to saturate each of the Grassmann integrations appearing in the fermionic measure $\mathrm{d}\mu_F\mathrm{d}\mu_f$. The integrals over the SUSY coordinates ξ_α and ξ'_α are saturated by the explicit insertion of the four fermion fields as given in (4.2). In the k instanton sector, there are also Grassmann integrations corresponding to the $4(k-1)$ remaining vector multiplet fermion zero modes and $2kN_f$ fermion zero modes from the hypermultiplets. These integrations can only be saturated by bringing down powers of the Grassmann quadrilinear term appearing in $\tilde{S}$. Clearly this only gives a non-zero result if the number of remaining zero modes from the vector and hyper-multiplets are equal. This requires that

$$4(k-1) = 2kN_f\,. \tag{5.3}$$

Thus we see that different instanton sectors contribute to the metric on the Coulomb branch for different values of N_f [1]. For $N_f = 0$, the above condition is trivially satisfied for $k = 1$. This corresponds to the one-instanton contribution in $N = 4$ SYM theory which was calculated in [6]. Similarly, for a single massless hypermultipet, (5.3) requires $k = 2$, which corresponds to a non-vanishing two instanton contribution that we will calculate below. For $N_f > 1$ there are no instanton corrections to the metric on the Coulomb branch.

We now specialize to the case $N_f = 1$, $k = 2$ where the remaining Grassmann integrals are saturated by bringing down two powers of the quadrilinear term in $\tilde{S}$ which gives

$$G^{(4)}(x_1,x_2,x_3,x_4) = 2^{15}\pi^3 M_W \Big(\frac{1}{8\pi^2}\int_{M_k} F\wedge F\Big) \exp(-2S_{\mathrm{cl}} + 2i\sigma)\, \times \tag{5.4}$$
$$\times \int \mathrm{d}^3X\, \epsilon^{\alpha'\beta'} S_{\mathrm{F}}(x_1-X)_{\alpha\alpha'} S_{\mathrm{F}}(x_2-X)_{\beta\beta'} \epsilon^{\gamma'\delta'} S_{\mathrm{F}}(x_3-X)_{\gamma\gamma'} S_{\mathrm{F}}(x_1-X)_{\delta\delta'}\,,$$

where F is a two-form on the relative moduli space of two instantons M_2 which is constructed from the components of curvature tensor of the $O(2)$ bundle which appears in (4.10). Choosing coordinates Y^a on M_2, with $a = 1,2,3,4$, we have,

$$F \equiv \tfrac{1}{2}F^{(12)}_{ab}\mathrm{d}Y^a \wedge \mathrm{d}Y^b\,. \tag{5.5}$$

The moduli space in question is exactly the AH manifold. We must also include the effect of the Z_2 symmetry acting on the monopole moduli space. It's action on the centre of charge is $\theta \to \theta + \pi$. It simultaneously acts on the ψ coordinate on the AH manifold (see (5.11) below) as $\psi \to \psi + \pi$. As θ does not appear in the integrand, we use it to restrict the range of ψ to 2π. Fortunately the integral of $F \wedge F$ over the AH manifold has been evaluated explicitly in [22, 24] where this integral appears as part of the volume contribution to the index of the Dirac operator. Explicitly, we find,

$$\frac{1}{8\pi^2}\int_{M_2} F \wedge F = \frac{1}{8}\,. \tag{5.6}$$

The resulting four-point function corresponds to an instanton-induced vertex in the low-energy effective lagrangian of the form

$$\mathcal{L}_{4F} = \kappa\bar{\lambda}^2\bar{\psi}^2 \exp\left(-2S_{\rm cl} + 2i\sigma\right), \tag{5.7}$$

with

$$\kappa = 2^{10}\pi^3 M_W \left(\frac{2\pi}{e^2}\right)^4 . \tag{5.8}$$

The four powers of $2\pi/e^2$ reflect the normalisation of the fermion kinetic terms.

All that remains is to compare the result of our calculation with the prediction of Seiberg and Witten [1]. We will be brief as the comparison is almost identical to that performed for the $N_f = 0$ case in [6]. As explained in section 3, $N = 4$ SUSY dictates that the exact effective action has the form (3.6). Hence, to obtain an exact solution of the low-energy theory one must specify the hyper-Kähler metric on the target space $\mathcal{M}$. Seiberg and Witten have conjectured that $\mathcal{M}$ is the double cover of the AH manifold which we have already met in an apparently unrelated context as the moduli space of two instantons. Hence the relevant metric is the one constructed explicitly by Atiyah and Hitchin and described in detail in [5].

The Atiyah-Hitchin manifold admits an $SO(3)$ isometry and hence the metric can be written in the form [27]

$$g_{ab}{\rm d}X^a{\rm d}X^b = f^2(r){\rm d}r^2 + a^2(r)\sigma_1^2 + b^2(r)\sigma_2^2 + c^2(r)\sigma_3^2\,. \tag{5.9}$$

Here σ_i, $i = 1, 2, 3$ are the three left-invariant one forms on the $SO(3)$ orbit parametrized by Euler angles θ, ϕ and ψ with ranges, $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$ and $0 \le \psi < 2\pi$,

$$\begin{aligned}\sigma_1 &= -\sin\psi\ {\rm d}\theta + \cos\psi\sin\theta\ {\rm d}\phi\\ \sigma_2 &= \cos\psi\ {\rm d}\theta + \sin\psi\sin\theta\ {\rm d}\phi\\ \sigma_3 &= {\rm d}\psi + \cos\theta\ {\rm d}\phi\,.\end{aligned} \tag{5.10}$$

Its double cover has an $SU(2)$ isometry with $0 < \psi < 4\pi$, the functions a, b and c are still the same. It is convenient to define cartesian coordinates for the three non-compact directions,

$$X = r\sin\theta\cos\phi \qquad Y = r\sin\theta\sin\phi \qquad Z = r\cos\theta\,, \tag{5.11}$$

and also to define complex coordinates;

$$z_1 = \frac{1}{\sqrt{2}}(X - iY) \qquad\qquad z_2 = \frac{1}{\sqrt{2}}(Z - i\psi)\,. \tag{5.12}$$

In this basis, one can show that the leading exponential correction in the Riemann tensor only appears in R_{1212} and its complex conjugate [6]. The function $f(r)$ depends on the exact definition of the radial parameter r. Following [26], we choose $f = -b/r$.[5] The hyper-Kähler condition forces the remaining components $a(r)$ $b(r)$ and $c(r)$ to obey a set of three non-linear ODE's which were analysed in [5]. After identifying the appropriate boundary conditions, explicit forms can be found for $a(r)$, $b(r)$ and $c(r)$ in terms of elliptic functions. For the present purposes we only require the large distance asymptotic forms of these components,

$$a^2 = r^2(1-\frac{2}{r})-4r^2e^{-r}+\ldots \quad b^2 = r^2(1-\frac{2}{r})+4r^2e^{-r}+\ldots \quad c^2 = 4(1-\frac{2}{r})^{-1}+\ldots . \tag{5.13}$$

Further corrections are supressed by powers of $1/r$ or $\exp(-r)$.

At this point we can substitute (5.9), with the above asymptotic forms for the components a, b and c, for the metric in the effective action (3.6). In particular, we can compare the power-law terms with the results of our perturbative calculation. This comparison yields the one-loop identifications, $(X, Y, Z) = (2S_{\rm cl}/M_W)(\phi_1, \phi_2, \phi_3)$ and $\psi = \sigma$. The first equality implies that $r = 2S_{\rm cl} + O(1/S_{\rm cl})$ (see footnote below). In the following it is important that this relation is correct to the order shown. In particular, one may verify explicitly that the additive constant which could in principle appear at one-loop on the RHS of the relation vanishes. At this order, the metric can be written in the form (3.7) with $U = 1 - 2/r$. The second equality implies that we are indeed dealing with the double cover instead of the single cover of AH (in which case we had $\psi = \sigma/2$ [6]).

A similar comparison of the fermion kinetic term for the fermions Ω^a in (3.6) with its counterpart in the one-loop effective action allows us to make the following identifications in the complex basis

$$\Omega^1 = \frac{2S_{\rm cl}}{M_W}\bar{\lambda} \qquad \Omega^2 = \frac{2S_{\rm cl}}{M_W}\bar{\psi}\,, \tag{5.14}$$

and similar for the conjugated fermions. With the above identifications, the exact effective action gives rise to a vertex which, just like (5.7), couples four fermions of the same chirality:

$$\mathcal{L}_{4F} = \frac{1}{4}K\left(\frac{2S_{\rm cl}}{M_W}\right)^4 R_{1212}\bar{\lambda}^2\bar{\psi}^2 + \text{h.c.} \tag{5.15}$$

Hence, to complete the comparison, we must use the explicit asymptotic form (5.13) of the metric to compute the leading exponentially suppressed contribution to R_{1212}. The relevant component of the Riemann tensor was computed to the required order in [6]. Using the relation between the coordinates and the massless fields up to one-loop, we have

$$R_{1212} = 8S_{\rm cl}e^{-2S_{\rm cl}+2i\sigma}\,. \tag{5.16}$$

[5]This approach differs slightly from [6] where the relationship $r = S_{\rm cl}$ was taken to define the radial coordinate and hence f. Fixing $f = -b/r$ means that the relationship between r and $S_{\rm cl}$ will receive quantum corrections.

Substituting the above result for R_{1212} in (5.15) we find precise agreement with the instanton induced vertex (5.7). The conjugated term will be generated by the corresponding two-anti-instanton process. This confirms the conjecture of Seiberg and Witten that the quantum Coulomb branch of the theory is the double cover of the Atiyah-Hitchin manifold.

In fact, as in [6], the result of our instanton calculation is actually sufficient to deduce this correspondence from first principles. That the exact Coulomb branch metric has the form (5.9) is a consequence of the $SU(2)_N$ R-symmetry of the theory. As mentioned above, the hyper-Kähler condition, which is necessary for $N = 4$ SUSY, leads to a set non-linear ODE's for the metric components $a(r)$, $b(r)$ and $c(r)$. A simple analysis of these equations given in [6] shows that one-loop and two-instanton data are enough to fix the boundary conditions and specify a unique solution.

In [1], Seiberg and Witten also proposed that introducing a mass for the hypermultiplet in this theory corresponds to Dancer's deformation of the double-cover of AH manifold. As we discussed in section 3, our perturbative results agree with this proposal. Unfortunately, there are no explicit formulae available for the the exponentially suppressed corrections to the asymptotic metric on the deformed manifold and thus it is not possible to perform a non-perturbative check in the massive case. On the other hand, we could instead assume that the proposal is correct and use the instanton techniques to predict the metric, along the lines of [28].

There is one feature of the massive theory is which is straightforward to check: the RG flow to the $N = 4$ theory without matter as the hypermultiplet mass is taken to infinity. Briefly, the mass term lifts the hypermultiplet zero modes and therefore allows a non-zero one-instanton contribution. Mass dependence is also introduced in the one-loop prefactor R_H as per equation (4.14). One may easily check that, in the decoupling limit, this contribution reproduces the one-instanton effect of the $N_F = 0$ theory which was calculated in [6].

6. Instanton effects in $N = 2$ theories

We now turn to the theory with $N = 2$ supersymmetry. We will initially specialize to the case of N_f massless hypermultiplets in the fundamental representation. As discussed in section 3, $N = 2$ supersymmetry allows the generation of a superpotential which leads to fermion bilinear terms in the effective action. To determine the instanton contribution to the superpotential we therefore calculate the large-distance behaviour of the the two-point correlator,

$$G^{(2)}(x_1, x_2) = < \lambda_\alpha(x_1)\lambda_\beta(x_2) > \ . \tag{6.1}$$

Collecting all the relevant factors, the k-instanton contribution to this correlator can be written as

$$G^{(2)}(x_1, x_2) = \int \mathrm{d}\mu_B \mathrm{d}\nu_F \mathrm{d}\mu_f \, \lambda_\alpha^{\mathrm{LD}}(x_1)\lambda_\beta^{\mathrm{LD}}(x_2)\, S_V R_H \exp\left(-\tilde{S}_{N=2}\right), \tag{6.2}$$

where the various quantities appearing in the above expression are defined in equations (4.1), (4.2), (4.17), (4.9), (4.19), (4.18) and (4.14).

To obtain a non-zero contribution, all the Grassmann integrations appearing in the fermionic part of the measure must be saturated. As in the $N = 4$ case, the modes which correspond to supersymmetry transformations of the instanton are saturated by the massless fermion fields given in (4.2). The remaining fermion zero modes comprise $2(k-1)$ modes from the vector multiplet each with $U(1)_N$ charge $+1$ and $2kN_f$ modes from the massless hypermultiplets with charge -1. As these modes can only be lifted in $U(1)_N$ neutral pairs, the condition for a non-zero contribution is

$$2(k-1) = 2kN_f\,. \tag{6.3}$$

Clearly this condition can only be satisfied for $k = 1$ and $N_f = 0$. Thus we will calculate a one instanton effect in the $N = 2$ without matter couplings.

Performing the integration over bosonic and fermionic zero modes of the one instanton solution, we determine

$$G^{(2)}(x_1, x_2) = \frac{2^9\pi^3}{e^2} M_W^3 \exp(-S_{\rm cl} + i\sigma) \int \mathrm{d}^3x \epsilon^{\alpha'\beta'} S_{\mathrm{F}\alpha\alpha'}(x_1 - x) S_{\mathrm{F}\beta\beta'}(x_2 - x)\,. \tag{6.4}$$

This correlator corresponds to a two-point vertex in the low-energy effective action of exactly the type expected from the general expression (3.14). By comparing the fermion kinetic term for the massless fermion λ with the kinetic term of the fermion Ψ in (3.11) one finds that these fermions are related as $\lambda = (M_W/S_{\rm cl})\bar{\Psi}$. Comparing the contribution of the fermion bilinear vertex in (3.14) to $G^{(2)}$ with (6.4) we can determine the superpotential,

$$W(Z) = \frac{e^4 S_{\rm cl}^3}{2^4\pi^5} \exp(-S_{\rm cl} + i\sigma) = \frac{e^2}{2^4\pi^5} \exp(-Z)\,, \tag{6.5}$$

where Z is the complex coordinate on the Coulomb branch given at one-loop by (3.12). This is the superpotential first obtained by Affleck, Harvey and Witten in [8] although the calculation of one-loop fluctuations around the instanton background included above is new as is the overall normalisation of the superpotential. In particular, the prefactor of $S_{\rm cl}^3$ in (6.5) which comes from the one-loop determinants precisely reflects the fact that W is holomorphic with respect to the one-loop corrected complex structure.

In the case of massive hypermultiplets the story is more complicated. The complex mass term explicitly breaks the $U(1)_N$ symmetry and hence the condition (6.3) for a non-zero contribution no longer applies. In particular, when each complex mass is non-zero, there are one-instanton contributions to the superpotential for each value of N_f. As all the hypermultiplet zero modes are lifted, we must omit the factor $d\mu_f$ in (6.2). In the massive case we must also use the definition (4.15) for the one-loop factor

R_H. The resulting superpotential is:

$$
\begin{aligned}
W(Z) &= \frac{e^4 S_{\rm cl}^3}{2^4\pi^5}\left(\prod_{i=1}^{N_f}\left[\frac{\sqrt{|m_i|^2+(\tilde m_i+\frac{1}{2}M_W)^2}+\tilde m_i+\frac{1}{2}M_W}{\sqrt{|m_i|^2+(\tilde m_i-\frac{1}{2}M_W)^2}+\tilde m_i-\frac{1}{2}M_W}\right]\right)^{-\frac{1}{2}}\exp(-S_{\rm cl}+i\sigma) \\
&= \frac{e^4}{2^4\pi^5}\exp(-Z)\,, \qquad (6.6)
\end{aligned}
$$

where we have used (3.13) to express W as a holomorphic function of the complex field Z. This result agrees with the proposal for the exact superpotential made in [9] up to the overall normalisation which is not explicitly specified in that reference. When expressed as a function of Z, we see that the one-instanton contribution to the superpotential has no dependence on the number of flavors or their masses.[6] This dependence has been completely absorbed in the definition of Z, or, in other words, in the complex structure. In principle there can also be multi-instanton corrections in the massive case, however we will not attempt to calculate them here.

Finally, we note that we may flow from the $N=4$ to the $N=2$ theory by the addition of a massive half-hypermultiplet in the adjoint representation. As in the fundamental case, in order to lift the extra fermionic adjoint zero modes, we require either $M>0$ or $M_W<|\tilde M|$. The k-instanton contribtution to the two fermi correlator, $G^{(2)}$, is equal to the k-instanton contibution to the four-fermi correlator, $G^{(4)}$ in the theory with full $N=4$ supersymmetry. More precisely, the prefactor to $G^{(2)}$ is equal to that of $G^{(4)}$ multiplied by the determinant (4.16) and the factor $M(2M_W/M)^k$. This combination has the right behaviour in the decoupling limit $M\to\infty$.

Acknowledgments

We are grateful to J. de Boer, C. Houghton, V. Khoze, N. Manton, M. P. Mattis, A. Mountain and P. Sutcliffe for useful discussions. DT is grateful to the University of Washington for hospitality while this work was completed. The work of DT and SV was supported by PPARC.

Appendix A: Wilsonian effective action at one loop

In this appendix we calculate the one-loop contribution to the renormalisation of the coupling constant. The calculation is a direct generalisation of that performed in Appendix B of [6] and employs the background field method. We restrict ourselves initially to $N=4$ multiplets, discussing the $N=2$ case at the end. Representations of the $N=4$ supersymmetry algebra allow for either vector or hypermultiplets. The vector multiplet is always in the adjoint representation of the $SU(2)$ gauge group, while

[6] If $m_i=0$ and $M_W>2|\tilde m_i|$, the change of variables between Z and $S_{\rm cl}$ is ill-defined. In this case the superpotential vanishes.

hypermulitplets may be in either the adjoint or fundamental representation. Hypermultiplets may also have an arbitrary mass, $\vec{m}$, (see section 2). Integrating out the high momentum modes for any multiplet results in a term for the Wilsonian effective action containing the massless bosonic fields of the vector multiplet. This term is of the form

$$\tfrac{1}{2}C\int \frac{\mathrm{d}^3k}{(2\pi)^3}\{A_\mu(-k)A_\nu(k)(k^2g^{\mu\nu}-k^\mu k^\nu)+(A^\dagger(-k)A(k)+\phi(-k)\phi(k))k^2\}\,, \quad (6.7)$$

where we have dropped terms of $\mathcal{O}(k^4)$. C can be written in the general form

$$C = 2\sum_w\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{w^2\chi}{(p^2-|\vec{m}+\sqrt{2}w\vec{\mathrm{v}}|^2)^2} = \frac{1}{4\pi}\sum_w\frac{w^2\chi}{|\vec{m}+\sqrt{2}w\vec{\mathrm{v}}|}\,, \quad (6.8)$$

where $\chi=-1$ for a vector multiplet and $\chi=+1$ for a hypermultiplet and w are the weights of the representation; $w=-1/2,+1/2$ for the fundamental representation and $w=-1,0,1$ for the adjoint. Comparing to the tree-level low-energy effective action, we find a finite renormalisation to the coupling constant. For the $N=4$ theory with one massless vector multiplet, N_f fundamental hypermultiplets with mass $\vec{m}_i$, $i=1,...,N_f$ and N_a adjoint hypermultiplets with mass $\vec{M}_I$, $I=1,...,N_a$, the renormalisation is

$$\begin{aligned}\frac{2\pi}{e^2}\to\frac{2\pi}{e^2}-\frac{1}{2\pi M_W}+\frac{1}{2^4\pi}\sum_{i=1}^{N_f}\left(|\vec{m}_i+\vec{\mathrm{v}}/\sqrt{2}|^{-1}+|\vec{m}_i-\vec{\mathrm{v}}/\sqrt{2}|^{-1}\right)\\ +\frac{1}{4\pi}\sum_{I=1}^{N_a}\left(|\vec{M}_I+\sqrt{2}\vec{\mathrm{v}}|^{-1}+|\vec{M}_I-\sqrt{2}\vec{\mathrm{v}}|^{-1}\right). \quad (6.9)\end{aligned}$$

We mention a few special cases. For $N_a=0$ and $\vec{m}_i=0$, the behaviour of the renormalised coupling constant is "$2-N_f$". This contrasts the "$4-N_f$" behaviour in four-dimensional $\mathcal{N}=2$ $SU(2)$ super Yang-Mills theory. In [1] this was explained using an anomaly argument and noticing that in the background of the appropriate instanton the four-dimensional vector mutiplet fermions have twice as many zero modes as their three-dimensional counterparts due to extra super-conformal modes. See [29] for a related discussion.

The theory with $N=8$ supersymmetry is obtained by taking $N_f=0$, $N_a=1$ with $\vec{M}=0$. As expected, the coupling constant is not renormalised in this case.

Finally, we may view the $N=4$ theory as the $N=2$ theory with an additional adjoint half-hypermulitplet. Turning this around, the renormalised coupling constant for the pure $N=2$ theory is obtained from the $N=4$ theory by "subtracting" a massless adjoint half-hypermultplet. Formally inserting $N_f=0$ and $N_a=-1/2$, $\vec{M}=0$ into equation (6.9) yields

$$\frac{2\pi}{e^2}\to\frac{2\pi}{e^2}-\frac{3}{4\pi M_W}\,, \quad (6.10)$$

which is indeed the correct renormalisation of the pure $N=2$ coupling constant. We may further add adjoint or fundamental half-hypemultiplets to this theory by augmenting equation (6.10) with the last two terms of (6.9), resulting in a "$3-N_f$" behaviour for the renormalised coupling constant.

References

[1] N. Seiberg and E. Witten, in *The Mathematical Beauty of Physics*, edited by J. M. Drouffe and J.-B. Zuber, World Scientific, 1997, p. 333 [hep-th/9607163].

[2] G. Chalmers and A. Hanany, *Nucl. Phys.* **B 489** (1997) 223 [hep-th/9608105].

[3] N. Seiberg, *Phys. Lett.* **B 384** (1996) 81 [hep-th/9606017].

[4] A. Hanany and E. Witten, *Nucl. Phys.* **B 492** (1997) 152 [hep-th/9611230].

[5] M. Atiyah and N. Hitchin, *The Geometry and Dynamics of Magnetic Monopoles*, Princeton University Press, 1988.

[6] N. Dorey, V.V. Khoze, M.P. Mattis, D. Tong and S. Vandoren, *Nucl. Phys.* **B 502** (1997) 59 [hep-th/9703228].

[7] A. Dancer, *Comm. Math. Phys.* **158** (1993) 545; *Q. J. Math.* **45** (1994) 463.

[8] I. Affleck, J. Harvey and E. Witten, *Nucl. Phys.* **B 206** (1982) 413.

[9] J. de Boer, K. Hori, Y. Oz and Z. Yin, *Nucl. Phys.* **B 502** (1997) 107 [hep-th/9702154]; J. de Boer, K. Hori and Y. Oz, *Nucl. Phys.* **B 500** (1997) 163 [hep-th/9703100].

[10] O. Aharony, A. Hanany, K. Intrilligator, N. Seiberg and M.J. Strassler, *Nucl. Phys.* **B 499** (1997) 67 [hep-th/9703110].

[11] J. De Jaegher, B. de Wit, B. Kleijn, and S. Vandoren, *Nucl. Phys.* **B 514** (1998) 553 [hep-th/9707262].

[12] L. Alvarez-Gaumé and D.Z. Freedman, *Comm. Math. Phys.* **80** (1981) 443.

[13] N.J. Hitchin, A. Karlhede, U. Lindström and M. Roček, *Comm. Math. Phys.* **108** (1987) 535.
H. Pederson and Y.S. Poon, *Comm. Math. Phys.* **80** (1988) 569.

[14] C.J. Houghton, *Phys. Rev.* **D 56** (1997) 1220 [hep-th/9702161].

[15] G. Chalmers, *The implicit metric on a deformation of the Atiyah-Hitchin manifold*, hep-th/9709082.

[16] A.M. Polyakov, *Nucl. Phys.* **B 120** (1977) 429.

[17] N. Dorey, V.V. Khoze and M.P. Mattis, *Nucl. Phys.* **B 502** (1997) 94 [hep-th/9704197].

[18] C. Bernard, *Phys. Rev.* **D 19** (1979) 3013.

[19] E.J. Weinberg, *Phys. Rev.* **D 20** (1979) 936.

[20] C. Callias, *Comm. Math. Phys.* **62** (1978) 213.

[21] J. Polchinsky and P. Pouliot, *Phys. Rev.* **D 56** (1997) 6601 [hep-th/9704029].

[22] S. Sethi, M. Stern and E. Zaslow, *Nucl. Phys.* **B 457** (1995) 484 [hep-th/9508117].

[23] M. Cederwall, G. Ferretti, B.E.W. Nilsson and P. Salomonson, *Mod. Phys. Lett.* **A 11** (1996) 367 [hep-th/9508124].

[24] J.P. Gauntlett and J.A. Harvey, *Nucl. Phys.* **B 463** (1996) 287 [hep-th/9508156].

[25] N. Manton and B. Schroers, *Ann. Phys. (NY)* **225** (1993) 290.

[26] G. Gibbons and N. Manton, *Nucl. Phys.* **B 274** (1986) 183.

[27] G. Gibbons and C. Pope, *Comm. Math. Phys.* **66** (1979) 267.

[28] C. Fraser and D. Tong, *Instantons, three dimensional gauge theories and monopole moduli spaces*, to be published in *Phys. Rev.* **D**, hep-th/9710098.

[29] C. Boulahouache and G. Thompson, *One loop effects in various dimensions and D-branes*, hep-th/9801083.